TRINITY COLLEGE

Mathematics Methods Units 3/4
Test 2 2017

Section 1 Calculator Free

Applications of Calculus

STUDENT'S NAME Solutions

DATE: Tuesday 28 March **TIME:** 25 minutes **MARKS:** 27

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (9 marks)

Differentiate each of the following with respect to x. (Do not simplify your answers):

(a) $y = x^5 e^{-3x}$ [2]

$$\frac{dy}{dx} = x^5 . -3e^{-3x} + e^{-3x} . 5x^4$$

(b) $y = \cos\left(\sqrt{7+e^x}\right)$ [3]

$$= \cos\left((7+e^x)^{1/2}\right)$$

$$\frac{dy}{dx} = -\sin\left((7+e^x)^{1/2}\right) \times \frac{1}{2}(7+e^x)^{-1/2} \times e^x$$

(c) $y = f(5-3x)$ [2]

$$\frac{dy}{dx} = f'(5-3x) \times (-3)$$

(d) $y = \int_x^1 (1+2t)^2 \, dt$ [2]

$$= -\int_1^x (1+2t)^2 \, dt$$

$$\frac{dy}{dx} = -(1+2x)^2$$

2. (9 marks)

(a) Determine:

(i) $\int 2x + e^{-2x} + e\, dx$ [3]

$$= x^2 - \frac{1}{2}e^{-2x} + ex + c$$

(ii) $\int \frac{xe^{1-2x^2}}{2}\, dx$ [3]

$$= \frac{1}{2} \times \frac{1}{-4} \int -4x e^{1-2x^2}\, dx$$

$$= \frac{1}{-8} e^{1-2x^2} + c$$

(b) Evaluate $\int_1^{\pi} \frac{d}{dx}\left(\frac{\sin x}{x^2+1}\right) dx$ [3]

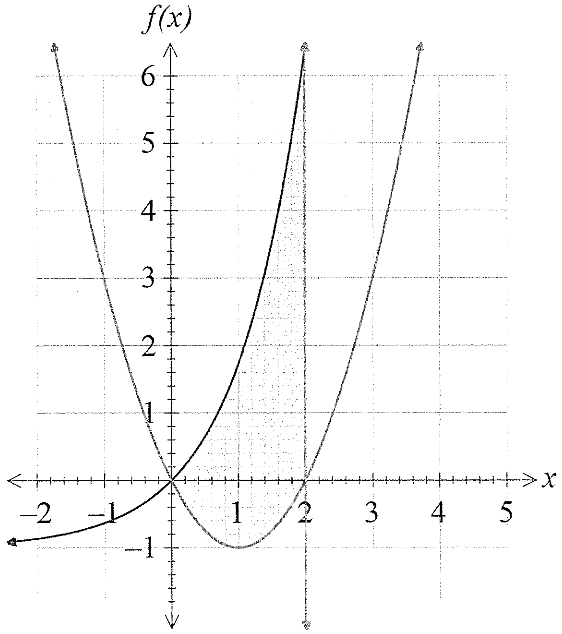
$$= \left[\frac{\sin x}{x^2+1}\right]_1^{\pi}$$

$$= \left(\frac{\sin \pi}{\pi^2+1}\right) - \frac{\sin 1}{1^2+1}$$

$$= -\frac{\sin 1}{2}$$

3. (5 marks)

Calculate the area enclosed between the functions $e^x - 1$, $x(x-2)$ and the line $x = 2$ as indicated on the graph below:



$$\text{Area} = \int_0^2 e^x - 1 - x(x-2) \; dx$$

$$= \int_0^2 e^x - x^2 + 2x - 1 \; dx$$

$$= \left[e^x - \frac{x^3}{3} + x^2 - x \right]_0^2$$

$$= \left(e^2 - \frac{8}{3} + 4 - 2 \right) - \left(e^0 - 0 + 0 - 0 \right)$$

$$= e^2 - \frac{8}{3} + 2 - 1$$

$$= e^2 - \frac{8}{3} + \frac{3}{3}$$

$$= e^2 - \frac{5}{3} \text{ units}^2$$

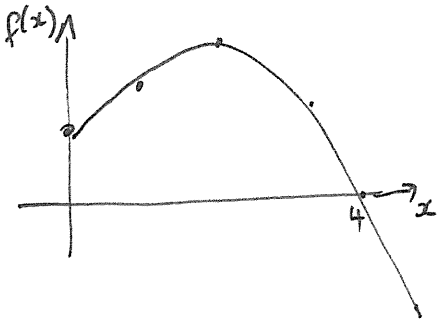
4. (4 marks)

A continuous function $f(x)$ is increasing on the interval $0 < x < 2$ and decreasing on the interval $2 < x < 5$. Some of its values are given in the table below:

x	0	1	2	3	4	5
$f(x)$	5	17	24	13	0	-29

The function $F(x)$ is defined, for $0 \le x \le 5$, by $F(x) = \int_0^x f(t)\,dt$.

(a) At which value of x in the interval $0 \le x \le 5$ is $F(x)$ greatest? Justify your answer. [2]



$F(x)$ is the area up to pt x

$\therefore$ $F(x)$ is maximum when reach the root

$\therefore$ $x = 4$ gives max $F(x)$

(b) At which value of x in the interval $0 \le x \le 5$ is $F'(x)$ greatest? Justify your answer. [2]

$$F'(x) = \frac{d}{dx} F(x)$$

$$= \frac{d}{dx} \int_0^x f(t)\,dt$$

$$= f(x)$$

$\therefore$ $F'(x)$ is max when $f(x)$ is max and $f(x)$ is increasing on interval $0 \le x < 2$

$\therefore$ $F'(x)$ is greatest at $x = 2$

TRINITY COLLEGE

Mathematics Methods Units 3/4
Test 2 2017

Section 2 Calculator Assumed
Applications of Calculus

STUDENT'S NAME ______________________________

DATE: Tuesday 28 March **TIME:** 25 minutes **MARKS:** 25

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser
Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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5. (5 marks)

During a volcanic eruption a rock is ejected from the top of the volcano. The rock rises upward and then falls onto a flat plain 1500 metres below the top of the volcano. During its flight, the vertical velocity of the rock, v m/s, is given by

$$v = 160 - 9.8t$$

Where t seconds is the time after the ejection of the rock

(a) How high does the rock rise above the top of the volcano? [3]

$v = 160 - 9.8t$

$x = -4.9t^2 + 160t + \underbrace{\cancel{c}}_{=0}$ (calculator)

Using calc, tp $(16.33, 1306.12)$

$\therefore$ reaches 1306 m above top of volcano

(b) How long does it take for the rock to reach the plain below? [2]

Solve $-1500 = -4.9t^2 + 160t$

$\Rightarrow t = \cancel{-7.6}, 40.3$

$\therefore$ rock reaches ground after 40.3 sec

6. (6 marks)

A radioactive substance is decaying exponentially, according to the formula

$$A(t) = A_0 e^{-kt} \text{ , where } A(t) \text{ kg is the amount at time } t \text{ years.}$$

(a) Determine k, correct to 4 decimal places, given that the half-life of the substance is 12 years. [2]

$\Rightarrow \quad \frac{1}{2} = e^{-k(12)}$

$\Rightarrow \quad k = 0.0578$

A second radioactive substance is also decaying exponentially, according to the formula

$$B(t) = B_0 e^{-0.04t} \text{ , where } B(t) \text{ kg is the amount at time } t \text{ years.}$$

(b) Which of these substances is decaying faster? Justify your answer briefly. [1]

A(t) is decaying faster as it has a larger -ve exponent

At a certain location there was exactly the same amount of these two substances at the beginning of the year 2017.

(c) In what year will the ratio of the amount of one of these substances to the other be $2:1$? [3]

In 2017, both have the same amount, C_0.

$\therefore \quad A(t) = C_0 e^{-0.0578t}$ and $B(t) = C_0 e^{-0.04t}$

A(t) is decaying faster, therefore ratio of 2 : 1 will be when A(t) is the smaller.

$\Rightarrow \quad 2\cancel{C_0} e^{-0.0578t} = \cancel{C_0} e^{-0.04t}$

$\Rightarrow \quad t = 39.02$ yrs

$\therefore$ At begining of 2056

7. (7 marks)

The rate of population change of a bacteria culture is modelled by $\frac{dP}{dt} = 100e^{-0.01t}$ where t is in hours.

(a) Determine the initial instantaneous rate of change of P with respect to t. [1]

$$\left.\frac{dP}{dt}\right|_{t=0} = 100 \text{ bac/hr}$$

(b) Describe the rate of change for large values of t. [1]

as $t \to \infty$, $\frac{dP}{dt} \to 0$

ie. $\frac{dP}{dt}$ gets closer and closer to 0

(b) Determine the net change in population during the first 10 hours. [2]

$$\text{net change} = \int_0^{10} 100e^{-0.01t}\,dt$$

$$= 951.63$$

$$\approx 952 \text{ bac}$$

(c) Determine the average change in population during the first 10 hours. [1]

$$\text{ave change} = \frac{952}{10}$$

$$= 95.2 \text{ bac / yr}$$

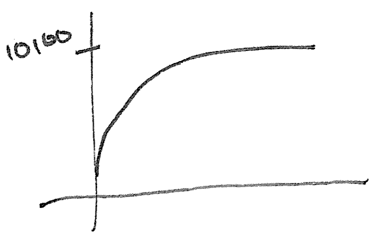
(d) Given that the initial population was 100, determine the maximum population size. Show clearly how you obtained your answer. [2]

$$\frac{dP}{dt} = 100e^{-0.01t}$$

$$\Rightarrow P = -10000e^{-0.01t} + c$$

$$\Rightarrow P = 10100 - 10000e^{-0.01t}$$

Plotting this



$\therefore$ Max population size is 10100 bacteria

8. (7 marks)

The acceleration, $a(t)\ m\,s^{-2}$, of an object moving in a straight line is given by:

$a(t) = At + B$, where A and B are non-zero constants.

The object is at rest initially and again after 10 seconds, and the object returns to its initial position after T seconds.

(a) Evaluate T [4]

$v(t) = \frac{1}{2}At^2 + Bt + C$ but $v(0) = 0$ $\therefore C = 0$

$v(10) = 0 \Rightarrow B = -5A$

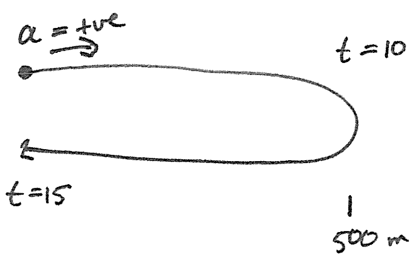
$x(t) = \frac{1}{6}At^3 + \frac{1}{2}Bt^2 + D$ but $x(0) = 0$ $\therefore D = 0$

$= \frac{1}{6}At^3 + \frac{1}{2}(-5A)t^2$

$= \frac{1}{6}At^2(t - 15)$

$\therefore$ body is at O again after 15 seconds

(b) Evaluate A and B given that the acceleration is positive initially and that the object travels a distance of 1 kilometre in the first T seconds. [3]



$\therefore$ Particle travels 500 m in first 10 seconds

$x(10) - x(0) = 500$

$\Rightarrow \frac{1}{6}A(10)^3 + \frac{1}{2}B(10)^2 = 500$ (1) but $B = -5A$ (2)

$\Rightarrow A = -6$, $B = 30$ } Use calculator to solve for eqn (1), (2)